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SINGULAR PERTURBATION METHODS AND THE WARM PLASMA MODEL

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ABSTRACT

It is often considered that taking into account the temperature, hence the compressibility, of the electron gas in a plasma is an improvement over the cold plasma model. This however leads to a number of peculiar and paradoxical results that show the need for some caution in applying this model. One result is that the boundary conditions for the warm plasma do not reduce to those for the cold plasma when the temperature approaches zero. Another is that evaluation of the impedance or the radiation from an antenna leads to widely different results according to the exact size of the antenna. This has led some authors to draw completely opposite conclusions as to the importance of acoustic waves. Both of these occurrences can be traced to the fact that the warm plasma equations are of higher order than those of the cold plasma and that the extra terms contain a small factor of the order of a/c, where a is the speed of sound and c that of light.

This suggests that the techniques of the singular perturbation theory can be applied to these problems. Typically the cold plasma model can be applied over wide ranges of the parameters (position, frequencies, angle of incidence) and only over narrow ranges forming so-called "boundary layers" does one need to use the warm plasma model. Then again simplified equations can be used and the solutions matched on both sides of the layer's boundary. A number of simple examples will illustrate this point of view. The analysis confirms that some results are highly sensitive to the values of some parameters: wire radius or gap size for an antenna, temperature of the medium, and incident angle of a plane wave. As a result, the corresponding

"resonances" cannot be observed in practice since the parameters will never be realized exactly enough. The boundary layer can then be neglected.

The description of the temperature effect by using the more exact kinetic theory is also discussed. The application of this theory is much more complicated, and in a few simple cases where it has been worked out, no significantly new result has been obtained.

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1. INTRODUCTION

Most studies of electromagnetic wave and antenna problems in plasmas are based on the cold plasma model which assumes that the electrons in the plasma have no thermal energy and form an incompressible fluid. In recent years, there have been many efforts to include the temperature effect of the plasma by using warm plasma models. This can be achieved in the frame of either the fluid description or the more exact kinetic theory. The warm plasma model based on kinetic theory is quite complicated mathematically and, therefore, has only been used in a few simple problems.

Because of the inclusion of the temperature effect, it is often believed that the warm plasma model is an improvement over the cold plasma model. However, this has led to a number of peculiar and paradoxical results that show the need for some caution in using this model. One result is that the field in the neighborhood of a rigid boundary for a low-temperature warm plasma is not a small perturbation of that for cold plasma, and consequently does not reduce to the cold plasma field when the temperature is reduced to zero. Another is the evaluation of the impedance or the radiation from an antenna. Widely different results are obtained depending on the exact size of the antenna. This has led authors to draw completely opposite conclusions as to the importance of the acoustic waves. Both of these occurrences can be traced to the following two facts:

(i) The (fluid) warm plasma equations are of higher order than those of cold plasma, and the extra term contains a factor

$$\delta = \frac{a}{c} \tag{1.1}$$

where a is the speed of sound and c that of light (in vacuum). Assuming the electrons in the plasma to form a perfect gas with 3 degrees of freedoms we have

$$a = \sqrt{3KT/m} \tag{1.2a}$$

or

$$\delta = 2.24 \times 10^{-5} \sqrt{T} \tag{1.2b}$$

where K is the Boltzman constant, m is the electron mass, and T is the absolute temperature of the electrons in degrees (Kelvin). For plasmas encountered in the ionosphere or in laboratories for microwave experiments, T rarely exceeds several thousand degrees. Therefore, the factor δ is a very small number.

(ii) The rigid boundary condition for the velocity $\bar{V}$ at the boundary with normal $\hat{n}$, i.e.,

$$\hat{n} \cdot \bar{V} = 0 \tag{1.3}$$

is enforced in warm plasma but relaxed in cold plasma. In the fluid description of plasma, the governing equations are the usual Maxwell equations plus the equation of motion for the electrons. For an isotropic plasma, the latter takes the form:

$$\text{cold plasma:} \quad \bar{V} = \frac{e\bar{E}}{i\omega m(1 + \nu/\omega)} \tag{1.4}$$

$$\text{warm plasma:} \quad \bar{V} + \delta^2 \frac{\nabla\nabla \cdot \bar{V}}{k_o^2 (1 + i\nu\omega)} = \frac{e\bar{E}}{i\omega m(1 + i\nu/\omega)} \tag{1.5}$$

where (-e), m, and ν are the charge, mass, and the collision frequency of the electron, respectively, and $k_o = \omega/c$. Note that (1.5) is a second-order partial differential equation for $\bar{V}$, while (1.4) is of zero order. In the limit T or $\delta \to 0$, the order of (1.5) is reduced

and, consequently, the boundary condition in (1.3) can no longer be satisfied.

The above two facts lead to the creation of the so-called "boundary layers" in the field solutions and suggest the application of the singular perturbation method to these problems.[1] Typically, the cold warm model can be applied over wide ranges of parameters (position, frequency, angle of incidence, etc.) and the result so obtained differs from the corresponding solution in the warm plasma model by terms of $O(\delta)$, which are negligible for all practical purposes. Only over narrow ranges of parameter forming boundary layers does one need the warm plasma model. Then, in the boundary layers, simplified equations can also be used and solutions matched on both sides of the layer's boundary.

In this paper we will consider a few simple examples to illustrate this point of view. To avoid unnecessary mathematical complications, the plasma is assumed to be isotropic throughout. In Section 2, the singular perturbation method is applied to a textbook-type problem for the purpose of demonstrating the formation of a boundary layer for the acoustic wave in the immediate neighborhood of a rigid boundary. Inside the layer, the acoustic wave contributes significantly to the total field so that the latter may satisfy the boundary condition in (1.3). Outside the layer, it suffers heavy attenuation due to the collision loss in the plasma and, therefore, can be safely ignored. The singular perturbation method uses this fact specifically and, therefore, may simplify the mathematical manipulations in the warm plasma model when applied to sophisticated problems.

In many boundary value problems in plasmas, we often are

interested in physical quantities such as the reflection coefficient, power ratio, impedance, etc. In Section 3, we show with two examples that these quantities obtained in a (low-temperature) warm plasma are nearly the same as those in cold plasma except in boundary layers in parameter space. In these boundary layers, the warm plasma solutions vary so rapidly in parameter space that they cease to be physically meaningful quantities. An attempt to improve the situation is to use a more sophisticated model to describe the temperature effect in the plasma. Thus, in Section 4, we turn to the kinetic theory. However, at least in the example we considered (namely, the reflection from a plasma half-space), the same boundary layer exists in both the fluid and the kinetic theory description!

In addition to the boundary value problems, another class of important plasma problems is the radiation from a prescribed current source $\bar{J}(\bar{r})$. An interesting question is whether the fields computed from cold plasma and warm plasma (fluid or kinetic theory) are significantly different (an indication of the importance of acoustic waves). This subject will be discussed in Section 5. The conclusion is that as long as $\bar{J}(\bar{r})$ is smooth enough this difference is negligibly small, otherwise any sensational results are possible.

2. ATTENUATION OF THE ACOUSTIC WAVE

The problem under consideration is the reflection of a plane wave from an infinite conducting plane in an isotropic lossy warm plasma (Figure 1). Let the incident field be an electromagnetic wave with

$$H_y^{(i)} = e^{-ik_o \sqrt{\varepsilon_1} \cos\theta_o x} \left[e^{ik_o \sqrt{\varepsilon_1} \sin\theta_o z - i\omega t}\right] \tag{2.1}$$

where $k_o = \omega/c$, $\varepsilon_1 = 1 - (\omega_p/\omega)^2(1 + i\nu/\omega)^{-1}$, $\mathrm{Im}\sqrt{\varepsilon_1} \geq 0$, ω_p = plasma frequency, and ν = collision frequency. The problem is to determine the scattered fields which satisfy the wave equations

$$\left(\frac{\partial^2}{\partial x^2} + k_o^2\, \varepsilon_1 \cos^2\theta_o\right) H_y = 0 \tag{2.2}$$

$$\left(\frac{\partial^2}{\partial x^2} + \frac{k_o^2\, \varepsilon_1 \tau^2}{\delta^2}\right) P = 0 \tag{2.3}$$

where

$$\tau = \sqrt{\left(1 + i\,\frac{\nu}{\omega}\right) - \delta^2 \sin^2\theta_o} \tag{2.4}$$

$$\delta = \frac{a}{c} = \text{ratio of sound and light speeds} \tag{2.5}$$

P = perturbed pressure in plasma

and the appropriate boundary conditions at $x = 0$ and $x \to \infty$. Since the scattered fields will have the same z-variation and time convention as the incident one, the common factor as appeared in [ ] in (2.1) for all

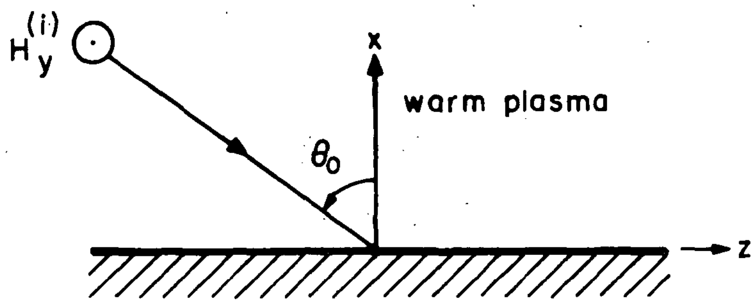

Figure 1. Reflection of a plane wave from a conducting plane in a lossy warm plasma

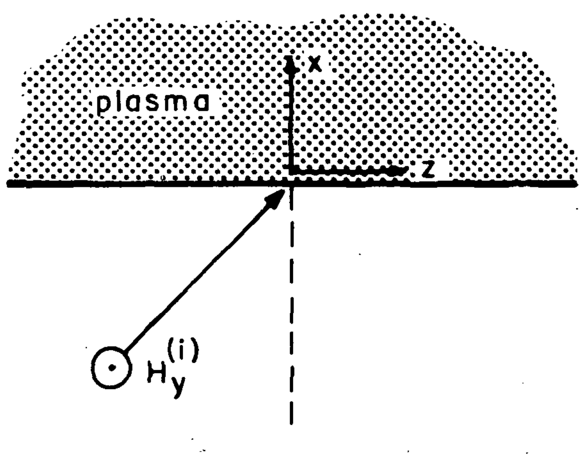

Figure 2. Reflection of a plane wave from a plasma half space

field quantities will be dropped hereafter.

This simple problem, of course, has an exact solution which can be obtained easily. However, in order to demonstrate the existence of a boundary layer in the scattered field (as $\delta \to 0$) in a convenient manner, we will attack this problem by a singular perturbation technique.[1] The basic idea of this technique lies in obtaining two asymptotic expansions for the scattered fields for the smallness of δ. One of the expansions is valid in a very small region close to the conducting plane, known as the boundary layer; the other is outside of the layer. By matching these two expansions at an overlapped region, we may determine certain constants that appear in the expansions and finally derive an expression uniformly valid inside as well as outside the boundary layer. In the present simple problem our uniformly valid asymptotic expression is actually the exact solution.

To apply the singular perturbation method, first let us consider the field expansion valid inside the boundary layer which is associated with the limit

$$\tilde{X} = \frac{k_o x}{\delta} \text{ fixed, } \delta \to 0. \quad \text{(inner limit)} \tag{2.6}$$

In terms of the new variable, the two wave equations in (2.2) and (2.3) become

$$\left[\frac{\partial^2}{\partial \tilde{X}^2} + \delta^2 (\varepsilon_1 \cos^2 \theta_o) \right] H_y = 0 \tag{2.7}$$

$$\left(\frac{\partial^2}{\partial \tilde{X}^2} + \varepsilon_1 \tau^2 \right) P = 0 \quad . \tag{2.8}$$

Consider (2.7) first. We are looking for an asymptotic solution of the form

$$H_y \sim f_o(\tilde{X}) + \delta f_1(\tilde{X}) + \delta^2 f_2(\tilde{X}) + 0(\delta^3) \quad . \tag{2.9}$$

Substituting (2.9) into (2.7) and equating the terms of same order of magnitude, this procedure results in

$$\frac{\partial^2 f_o}{\partial \tilde{X}^2} = \frac{\partial^2 f_1}{\partial \tilde{X}^2} = 0$$

$$\frac{\partial^2 f_2}{\partial \tilde{X}} + (\varepsilon_1 \cos^2\theta_o) f_o = 0 \quad .$$

When the above differential equations are solved and their results substituted into (2.9), we have

$$H_y \sim (C_1 + C_2\tilde{X}) + \delta(C_3 + C_4\tilde{X}) - \delta^2(\varepsilon_1\cos^2\theta_o)(C_1 \frac{\tilde{X}^2}{2} + C_2 \frac{\tilde{X}^3}{6}) + 0(\delta^3) \quad . \tag{2.10}$$

The constants, C's, will be determined later when the boundary conditions are applied. The wave equation for the acoustic wave in (2.8) is a regular one with no appearance of the perturbation parameter δ, and its solution is simply

$$P = Be^{i\sqrt{\varepsilon_1}\,\tau} + De^{-i\sqrt{\varepsilon_1}\,\tau} \quad . \tag{2.11}$$

To determine a part of the constants in (2.10) and (2.11), we will now

apply the boundary conditions at $x = 0$, namely,

$$E_z^{(t)}\ (x = 0) = 0,\quad V_x^{(t)}\ (x = 0) = 0 \tag{2.12}$$

where the superscript (t) signifies the total field, the incident plus the scattered. In terms of H_y and P, the two conditions in (2.12) become

$$\left[\delta(\sqrt{\varepsilon_1}\ q\ \sin\theta_o\ P + i\sqrt{\varepsilon_1}\ \cos\theta_o) - \frac{\partial}{\partial \tilde{X}}\ H_y\right]_{\tilde{X}=0} = 0 \tag{2.13a}$$

$$\left[\delta\left(\frac{\omega_p}{\omega}\right)^2 \frac{\sqrt{\varepsilon_1}\ \sin\theta_o}{1 + i(\nu/\omega)}\ (1 + H_y) - q\ \frac{\partial}{\partial \tilde{X}}\ P\right]_{\tilde{X}=0} = 0 \tag{2.13b}$$

where: $q = \dfrac{e}{\omega m(1 + i\nu/\omega)}$

e = magnitude of the charge of an electron

m = mass of an electron

Substitution of (2.10) and (2.11) into (2.13) leads to

$$C_2 = C_3 = 0 \tag{2.14a}$$

$$\delta \sin\theta_o\ \left(\frac{\omega_p}{\omega}\right)^2\ (1 + C_1) = iq\tau(1 + i\ \frac{\nu}{\omega})(B + D) \tag{2.14b}$$

$$1 - \frac{C_4}{i\sqrt{\varepsilon_1}\ \cos\theta_o} = iq(B - D)\tan\theta_o \tag{2.14c}$$

which are four conditions for the six undetermined constants.

Next, we will consider the field expansion valid outside the boundary layer which is associated with the limit

$$X = k_o x \text{ fixed}, \ \delta \to 0 \quad . \quad \text{(outer limit)} \tag{2.15}$$

In terms of the new variable X, the wave equations in (2.2) and (2.3) take the form

$$\left(\frac{\partial^2}{\partial X^2} + \varepsilon_1 \cos^2\theta_o \right) H_y = 0 \tag{2.16}$$

$$\left(\delta^2 \frac{\partial^2}{\partial X^2} + \varepsilon_1 \tau^2 \right) P = 0 \tag{2.17}$$

and the boundary condition is

$$H_y, \ P \to 0 \text{ as } X \to \infty \quad . \tag{2.18}$$

This solution of (2.16) and (2.18) is

$$H_y = A e^{i\sqrt{\varepsilon_1} \cos\theta_o X} \ , \quad \mathrm{Im}\sqrt{\varepsilon_1} > 0 \tag{2.19}$$

and that of (2.17) and (2.18) is

$$P \sim 0 \tag{2.20}$$

which means that the order of P is smaller than any algebraic power of δ.

Summarizing the results obtained so far, we have the inner expansion in (2.10) and (2.11) and the outer expansion in (2.19) and (2.20). There are a total of seven constants in A, B, D, and C's in these expansions, and four conditions in (2.14) for their determination. The final step in this method for solution involves the matching of these two expansions, which means roughly that the inner expansion as

$\tilde{X} \to \infty$ and the outer expansion as $X \to 0$ should be in agreement.

To carry out such a matching, we need to "stretch" the regions of validity for our inner and outer expansions so that they become overlapping. Introduce a new variable

$$X_\eta = \frac{X}{\eta} \quad \text{(fixed)} \tag{2.21}$$

where η is a function of the perturbation parameter δ, and is asymptotically larger than δ, explicitly

$$\delta,\ \eta \to 0,\ \text{but}\ (\eta/\delta) \to \infty \quad . \tag{2.22}$$

Note that $\tilde{X} = (\eta/\delta)X_\eta \to \infty$ and $X = \eta X_\eta \to 0$. Using limits in (2.21) and (2.22) in (2.10), we have

$$\text{Inner:} \quad H_y \sim C_1 + \eta(C_4 X_\eta) - \eta^2\left(\frac{\varepsilon_1 \cos^2\theta_o}{2}\, C_1 X_\eta^2\right) + O(\eta^3) \tag{2.23}$$

where (2.14a) has been used. Application of the same limit in (2.19) leads to

$$\begin{aligned}\text{Outer:} \quad H_y &= Ae^{\eta(i\sqrt{\varepsilon_1}\,\cos\theta_o\, X_\eta)} \\ &\sim A + \eta(i\sqrt{\varepsilon_1}\,\cos\theta_o\, AX_\eta) - \eta^2\left(\frac{\varepsilon_1 \cos^2\theta_o}{2}\, AX_\eta^2\right) \\ &+ O(\eta^3) \quad .\end{aligned} \tag{2.24}$$

Comparing (2.23) and (2.24), we have

$$C_1 = A,\ C_4 = i\sqrt{\varepsilon_1}\ A\cos\theta_o \quad . \tag{2.25}$$

The matching of P in (2.11) and (2.20) in a similar manner leads to

$$D = 0 \tag{2.26}$$

The three additional conditions in (2.25) and (2.26), together with those in (2.14), determine the two expansions completely. The final forms of the two expansions are

Inner: $$H_y \sim A\left[1 + \delta(i\sqrt{\varepsilon_1}\cos\theta_o\,\tilde{X}) + O(\delta^2)\right] \tag{2.27}$$

$$P = Be^{i\sqrt{\varepsilon_1}\,\tau\tilde{X}} \tag{2.28}$$

Outer: $$H_y = Ae^{i\sqrt{\varepsilon_1}\cos\theta_o\,X} \tag{2.29}$$

$$P \sim 0 \tag{2.30}$$

where: $$A = \frac{1 - \delta h}{1 + \delta h}\ , \quad B = \delta\frac{2h\cot\theta_o}{iq(1 + \delta h)} \tag{2.31}$$

$$h = \left(\frac{\omega_p}{\omega}\right)^2 \frac{1}{1 + i(\nu/\omega)}\,\frac{\sin\theta_o\tan\theta_o}{\tau}\ .$$

The inner expansion is valid when $\tilde{X}$ defined in (2.6) assumes a fixed value, and the outer expansion when X defined in (2.15) assumes a fixed value. From the results in (2.27) through (2.31), we may derive the expressions for other field components. In particular, we are interested in the normal component of the velocity, which is found to be

Inner: $$V_x \sim \frac{ik_oq\sin\theta_o}{\omega\varepsilon_o\sqrt{\varepsilon_1}}\,A\left[1 + \delta(i\sqrt{\varepsilon_1}\cos\theta_o\,\tilde{X}) + O(\delta)^2 - 2e^{i\sqrt{\varepsilon_1}\,\tau\tilde{X}}\right] \tag{2.32}$$

$$\text{Outer:} \qquad V_x \sim \frac{ik_o q \sin\theta_o}{\omega\varepsilon_o\sqrt{\varepsilon_1}} \, Ae^{i\sqrt{\varepsilon_1}\cos\theta_o X} \quad . \tag{2.33}$$

The final results in (2.27) through (2.33) will now be examined.

(i) Inside the boundary layer. The acoustic wave plays an important part in the total field solution, so as to insure the satisfaction of the boundary condition at $x = 0$. Away from the boundary at $x = 0$, the acoustic part decays exponentially as [see (2.32)]

$$\exp\left[-\tilde{X}\,\mathrm{Im}(\sqrt{\varepsilon_1}\,\tau)\right] = \exp\left[-k_o x \frac{\mathrm{Im}(\sqrt{\varepsilon_1}\,\tau)}{\delta}\right] \quad . \tag{2.34}$$

Thus, we may define a "skin depth," which is a measurement of the thickness of the boundary layer, namely,

$$x_o = \frac{\delta}{k_o\,\mathrm{Im}(\sqrt{\varepsilon_1}\,\tau)} = \frac{\delta}{k_o\,\mathrm{Im}\sqrt{1-\left(\frac{\omega_p}{\omega}\right)^2 + i\,\frac{\nu}{\omega}}} + 0(\delta^2) \quad . \tag{2.35}$$

At $x = x_o$, the acoustic wave is reduced by e^{-1} or 37 per cent of its magnitude at $x = 0$. If terms of $0(\delta^2)$ are dropped, the skin depth is independent of the incident angle θ_o, and, in general, is a very small number. As a numerical example, we have computed x_o for three typical sets of parameters encountered in the ionosphere. They are displayed in Tables I and II. In Table II, we note that there exists a low-frequency and a high-frequency limit

$$x_o = \begin{cases} a/\omega_p\ , & \omega \to 0 \\ 2a/\nu\ , & \omega \to \infty \end{cases} \tag{2.36}$$

which are independent of ω. The numerical data reveal that the skin depth for the acoustic wave is very small except for one case discussed below. At high frequency in the F layer, the acoustic skin depth is

TABLE I

TYPICAL PARAMETERS IN IONOSPHERE

Layer	Height(km)	T	a(m/s)	ν	ω_p
D(day)	60	300^o	$1.16x10^5$	10^7	$6x10^5$
E(night)	90	200^o	$9.5x10^4$	$7x10^5$	10^6
F	300	2000^o	$3x10^5$	10^3	$6x10^7$

TABLE II

SKIN DEPTH OF ACOUSTIC WAVE x_o (IN METER)

ω	10^5	$6x10^5$	10^6	10^7	$6x10^7$	10^8	10^9
D	$1.4x10^{-1}$	$6.7x10^{-2}$	$5.4x10^{-2}$	$2.6x10^{-2}$	$2.3x10^{-2}$	$2.3x10^{-2}$	$2.3x10^{-2}$
E	$9.5x10^{-2}$	$1.1x10^{-2}$	*$1.6x10^{-1}$	$2.7x10^{-1}$	$2.7x10^{-1}$	$2.7x10^{-1}$	$2.7x10^{-1}$
F	$5x10^{-3}$	$5x10^{-3}$	$5x10^{-3}$	$5x10^{-3}$	*1.7	$4.8x10^2$	$6x10^2$
λ_o**	$1.9x10^4$	$3.1x10^3$	$1.9x10^3$	$1.9x10^2$	$3.1x10^2$	1.9x10	1.9

* at plasma frequency

** free space wavelength in meter

large because of the small collision loss [see (2.36)]. However, as long as $(\omega_p/\omega)^2 \ll 1$, the electromagnetic wave and the acoustic wave are practically uncoupled and hence the skin depth is no longer a meaningful quantity for measuring the importance of the acoustic wave.

(ii) Outside the boundary layer. The contribution from the acoustic wave becomes negligibly small (due to heavy attenuation), and the total field is nearly entirely made of the electromagnetic wave. The amplitude of the reflected electromagnetic wave for H_y is given by

$$A = A_o \frac{1 - \delta h}{1 + \delta h} \tag{2.37}$$

where h is defined in (2.31), and A_o is the amplitude when a cold plasma model is used. Unless the angle of incidence is nearly $(\pi/2)$*, A in (2.37) can be expanded as

$$A = A_o \left[1 - 2h\delta + O(\delta^2) \right] . \tag{2.38}$$

---

* Note that, for $\theta_o = (\pi/2) - \Delta$ with $\Delta \ll 1$, we have

$$A \sim \frac{1 - (\delta/\Delta)(\omega_p/\omega)^2/\sqrt{1 + i\nu/\omega}}{1 + (\delta/\Delta)(\omega_p/\omega)^2/\sqrt{1 + i\nu/\omega}}$$

If (δ/Δ) assumes a fixed value, the value of A may differ considerably from unity. Thus, as a function of θ_o, A may be regarded as having a boundary layer defined by $0 < (\frac{\pi}{2} - \theta_o) < \Delta$ in parameter space where the warm and cold plasma solutions do not agree in the low temperature limit. This point will be pursued further in Section 3.

Thus, A is slightly smaller than A_o. The decrease in A is due to the conversion to the acoustic waves which is heavily damped-out due to collision; however, one must realize that the amount of decrease is in the order of δ, and is negligibly small. Therefore, we may conclude that outside the boundary layer, the scattered field obtainable from a warm plasma model is practically identical to that from a cold plasma model.

(iii) Uniformly valid expression. Our inner and outer expansions may be combined to give an expression uniformly valid for all x. The standard procedure is to add these two expansions and substract out their common part. Take (2.32) and (2.33) as an example; the common part (cp) is

$$cp = 1 + \delta(i\sqrt{\varepsilon_1} \cos \theta_o \tilde{X}) + 0(\delta^2) \quad .$$

Thus, the uniformly valid expression for V_x is given by

$$V_x = \frac{ik_o q \sin\theta_o}{\omega\varepsilon_o \sqrt{\varepsilon_1}} \quad A \left(e^{i\sqrt{\varepsilon_1} \cos\theta_o X} - 2e^{i\sqrt{\varepsilon_1} \tau\tilde{X}} \right) \quad . (2.39)$$

Not surprisingly, (2.39) turns out to be the exact solution for the present problem. We emphasize the fact that in problems where the exact solution is not available, the singular perturbation procedures as illustrated above may often be useful.

(iv) Lossless warm plasma. As may be seen from (2.35), in the case of $\omega > \omega_p$, the formation of a boundary layer lies in the inclusion of collision loss in the plasma model. Even though collision should be inevitably present in a realistic plasma, many plasma problems are investigated by using a lossless model. The absence of a loss mechanism in such an idealized model leaves the acoustic wave unattenuated when

$\omega > \omega_p$, and consequently it plays an integral part in the total field solution everywhere. Take V_x in (2.39) as an example. When the distance along x is measured in terms of the usual free space wavelength λ_o, the total field solution has an extremely rapid-varying part due to the unattenuated acoustic wave. Thus, variation of a fraction of one per cent in (x/λ_o) may result in a significant change in V_x. This, of course, does not agree with our observations in plasma experiments. This brings out the point that in using a warm plasma model certain field solutions (e.g. V_x) may critically depend on the collision loss even though the collision is small. In these situations, the lossless idealization may not be a justified one.

3. BOUNDARY LAYER IN FIELD SOLUTIONS

The acoustic wave in plasma, as illustrated in the previous section, is heavily damped due to the collision and, therefore, cannot travel a great distance away from the boundary (or source). Then the next question of interest is that, because of the energy loss in the acoustic wave, what is the modification on the the field solution for optical waves? We will use two examples to illustrate the answer that, except for a narrow range of parameters (boundary layers), the field solution for the optical wave in warm plasma differs from that in cold plasma only by terms of order δ. Thus, for most plasma problems encountered in practice, the difference is negligibly small. However, for parameters falling inside the boundary layers, the field solution for the optical wave may be drastically modified and, furthermore, it may vary significantly over a small fraction of an optical wavelength. For example, the evaluation of the conductance of a delta-source excited cylindrical antenna in warm plasma depends on the magnetic field in a boundary layer. Consequently, the conductance may have significantly different values depending on the precise width of the feed gap.

In both of the examples to be presented below, we will not include the collision effect in the plasma for two reasons. First, the presence of a small loss modifies the boundary layer somewhat but does not change its basic structure. Secondly, in analyzing many boundary value problems, particularly the one of antenna impedance, it is traditionally based on a lossless model. By not including the collision here, we perhaps can illustrate better why a variety of different results on antenna impedance can be obtained depending on the precise width of the gap.

In the first example we considered the reflection of an incident H-wave ($\bar{H}$ normal to the plane of incidence) from a warm plasma half space (Figure 2). The reflection coefficient is found to be[2]

$$\Gamma = \Gamma_o M \quad . \tag{3.1}$$

Here Γ_o is the reflection coefficient for cold plasma,

$$\Gamma_o = \frac{\sqrt{\varepsilon - \sin^2\theta_o} - \varepsilon\cos\theta_o}{\sqrt{\varepsilon - \sin^2\theta_o} + \varepsilon\cos\theta_o} \quad , \tag{3.2}$$

and M is the modification term due to the introduction of the acoustic wave *

$$M = \frac{1 + \delta h_-}{1 + \delta h_+} \tag{3.3}$$

where

$$h_\pm = \frac{(1 - \varepsilon)\sin^2\theta_o}{(\sqrt{\varepsilon - \sin^2\theta_o} \pm \varepsilon\cos\theta_o)\sqrt{\varepsilon - \delta^2\sin^2\theta_o}}$$

$$\varepsilon = 1 - (\omega_p/\omega)^2 \quad .$$

Now let us examine M as a function of ε and $\sin^2\theta_o$. For a given δ and $\delta \to 0$, it may be shown that

$$M = 1 + O(\delta) \tag{3.4}$$

---

* To include the collision loss, simply make the following two replacements:

$$\varepsilon = 1 - \frac{(\omega_p/\omega)^2}{(1 + i\nu/\omega)} \quad , \quad \delta = \frac{a}{c}\frac{1}{\sqrt{1 + i\nu/\omega}} \quad .$$

for all $0 \leq \sin^2\theta_o \leq 1$ and $\varepsilon < 1$ except when

$$\sin^2\theta_o = \frac{\varepsilon}{1+\varepsilon} + \Delta \tag{3.5}$$

where Δ is a small number. With $\Delta = 0$, (3.5) gives the condition for total transmission in cold plasma ($\Gamma_o = 0$, but not Γ). We will now examine the expression for Γ in (3.1) under the condition of (3.5), that is, the boundary layer of the reflection coefficient. As indicated in Figure 3, there are two subdivisions depending on whether ε itself assumes a fixed value or a value comparable to δ.

(a) <u>ε is fixed</u>. One may show

$$\Gamma_o \sim \frac{(1+\varepsilon)(\varepsilon^2 - 1)}{4\varepsilon^2}\,\Delta$$

$$M \sim 1 + \left(\frac{\delta}{\Delta}\right)\frac{2}{1+\varepsilon}\left(\frac{\varepsilon}{1+\varepsilon}\right)^{3/2} .$$

Thus, the reflection coefficient for warm plasma is of $O(\varepsilon)$ when $(\delta/\Delta) \to \infty$, or of $O(\Delta)$ when $(\delta/\Delta) \to 0$. In either case, the difference between Γ and Γ_o is negligibly small.

(b) <u>ε is comparable to δ</u>. The situation is more complicated. Let us concentrate on the case of $\varepsilon = \delta^2$; δ and Δ are the variables. The main results of our study are summarized in Table III where the values of Γ_o, M, Γ, R, and η for different orders of (Δ/δ) are listed. The parameter η is the ratio of acoustic power in the plasma and the incident power from the free space. The numerical values in Table III indicate the extremely rapid variation of the field solutions in the boundary layer. For example, when $\sin^2\theta_o = \delta^2 = 10^{-5}$ (corresponding to $\Delta = \delta^4$; T = 2000° and $\delta = 10^{-3}$), the reflection coefficient Γ is

TABLE III

FIELD SOLUTIONS IN BOUNDARY LAYER

Δ	Γ_o	M	Γ	η
$0(\delta^5)$	0	∞	1/3	4/9
δ^4	-1	0	0	1
$0(\delta^3)$	1	1	1	0

zero and the total incident power is totally pumped into the acoustic waves (η = 1). Thus, it can be said that the warm plasma model gives a completely different result from that of the cold plasma (Γ_o = -1). However, if we vary the incident angle by 5×10^{-7} radians so that $\sin^2\theta_o = \delta^2 + \delta^3$ (corresponding to $\Delta = \delta^3$), the reflection coefficient Γ becomes unity, in agreement with the result in cold plasma. Thus, total transmission and total reflection are separated by merely 5×10^{-7} radians of the incident angle!

In conclusion, in the problem of reflection from a warm plasma half space, the field solution is practically the same as that obtainable from a cold plasma model except when the parameters (ε, θ_o) fall inside a boundary layer. As indicated in Figure 3 (not to scale), the boundary layer is a region around $\varepsilon = 1 - (\omega_p/\omega)^2 = 0$, having a dimension in the order of δ in the parameter space ($\varepsilon - \theta_o$ plane). The field solution for parameters in the boundary layer varies extremely rapidly, and, therefore, has a dubious physical significance.

In some electromagnetic boundary value problems, the physical

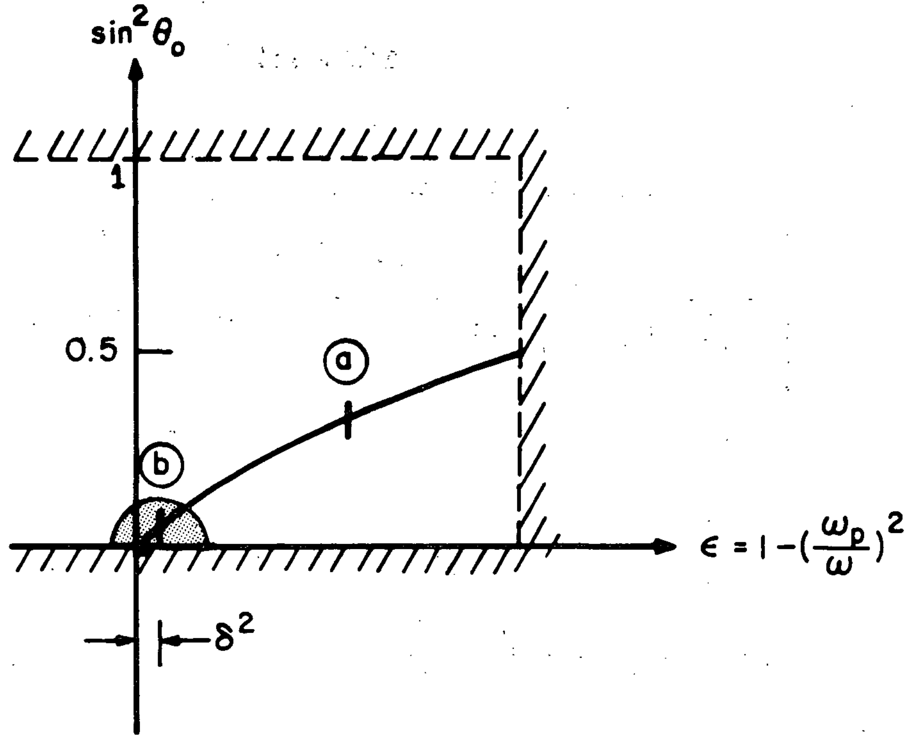

Figure 3. In the dotted region (not to scale), the cold and warm plasma solutions differ significantly for the problem sketched in Figure 2

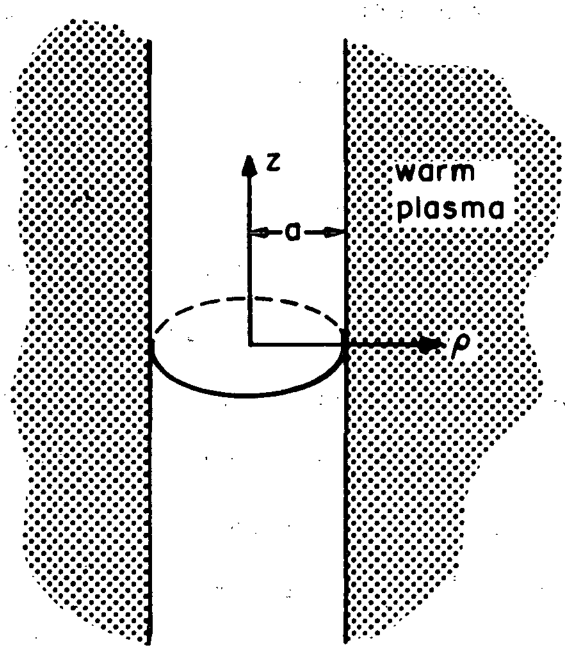

Figure 4. An infinitely long cylindrical antenna in a warm plasma

quantity of interest falls exactly within the boundary layer of the parameter space, and this may lead to a variety of different conclusions. Such situations will be illustrated in the second example, namely, the calculation of the impedance of an infinitely long cylindrical antenna immersed in an isotropic lossless warm plasma. The geometry of the antenna is shown in Figure 4; it is excited by a delta source with unit voltage amplitude located at $z = 0$. The expression for the induced current on the antenna has been obtained by a number of authors[3,4] and is duplicated below

$$I(z) = \frac{ik_e a}{120\pi} \int_{-\infty}^{\infty} \frac{e^{i\alpha z} d\alpha}{\left[\zeta_e \frac{H_o^{(1)}(\zeta_e a)}{H_1^{(1)}(\zeta_e a)} + \left(\frac{\omega_p}{\omega}\right)^2 \frac{\alpha^2}{\zeta_p} \frac{H_o^{(1)}(\zeta_p a)}{H_1^{(1)}(\zeta_p a)}\right]} \tag{3.6}$$

where $\zeta_n = \sqrt{k_n^2 - \alpha^2} = +i\sqrt{\alpha^2 - k_n^2}$, $n = e,p$

$k_e = k_o \sqrt{1 - (\omega_p/\omega)^2}$, $k_o = \omega/c$

$k_p = k_e/\delta$, δ = ratio of acoustic speed and light speed .

The integration contour in (3.6) is slightly above the real axis for $\alpha < 0$ and below for $\alpha > 0$. The admittance of the antenna is defined as

$$Y = G + iB = I(z = z_o) \tag{3.7}$$

where z_o is a suitably small distance from the idealized feed at $z = 0$ and may be identified with the half-gap width of the actual feed. We recall that even for an antenna situated in the free space, the imaginary

part of I(z) for z close to the gap has a logarithmic singularity, and, therefore, has a boundary layer. In order not to confuse that boundary layer (because of the delta source) with the boundary layer to be discussed below (because of the acoustic wave), we will concentrate on the real part of I(z) or the evaluation of the conductance G (for $\omega > \omega_p$). Furthermore, to facilitate the estimation of certain integrals, we will assume that the parameter

$$A = k_e a = k_o a \sqrt{1 - (\omega_p/\omega)^2}$$

is reasonably large so that $H_o^{(1)}(A)$ can be well-approximated by its first asymptotic term.* Summarizing, our task is to evaluate the real part of $I(z = z_o)$ given in (3.6) under the assumptions

$$\delta \to 0 \quad , \quad A >> 1 \quad . \tag{3.8}$$

There are two subdivisions, depending on the relative magnitude of z_o and the acoustic and optical wavelengths

$$\text{(i)} \quad \tilde{Z} = k_p z_o = \frac{k_o z_o}{\delta} \sqrt{1 - (\omega_p/\omega)^2} \quad \text{fixed (inner limit)} \tag{3.9}$$

$$\text{(ii)} \quad Z = k_e z_o = \frac{\tilde{Z}}{\delta} \quad \text{fixed (outer limit)} \quad . \tag{3.10}$$

We will first concentrate on the inner limit, where the gap width is so small that it has to be measured in terms of the acoustic wavelength. Under the assumptions in (3.8) and (3.9), the real part of $I(z_o)$ in (3.6) may be approximately evaluated with the result

$$G \sim G_o M \tag{3.11}$$

---

* The error is less than 5 per cent when A = 3.

Here G_o is the well-known (approximate) cold plasma solution for a thick antenna

$$G_o = \frac{k_e a}{120} \tag{3.12}$$

and M is the modification factor

$$M(\tilde{Z}) = 1 + 2\ (s^2 - 1)\ \cos(s\tilde{Z}) + \frac{2}{\pi}\int_0^{\pi/2} \frac{\cos(\tilde{Z}\sin\theta)}{(\omega/\omega_p)^2 + \tan^2\theta}\ d\theta$$
$$+ 0(\delta) \quad . \tag{3.13}$$

The second term in (3.13) is due to a surface wave contribution coming from a simple real zero of the denominator of the integrand in (3.6a). Under the condition in (3.8), the surface wave has a propagating constant (along z) approximately given by

$$\alpha \sim k_p s = k_p \sqrt{\frac{1}{1 - (\omega_p/\omega)^4}} \tag{3.14}$$

provided that $(\omega_p/\omega)^4$ is not too small. The third term in (3.13) comes from the portion of the integral in (3.6) in the range

$$k_e < \alpha < k_p \quad . \tag{3.15}$$

In the cold plasma model, there is no surface wave and there is no contribution to G from the integral range $k_e < \alpha$ (invisible range). Thus, the deviation of $M(\tilde{Z})$ from unity is the special feature of the warm plasma model. The following observation is made regarding $M(\tilde{Z})$ in (3.13).

(i) M does not reduce to unity as $\delta \to 0$. Thus when the gap width z_o falls inside the boundary layer [i.e. (3.9)], the cold plasma solution G_{cold} cannot be recovered from the warm plasma solution G by letting the temperature (or δ) go to zero. The same conclusion holds even if

a small loss in the medium is introduced.

(ii) M is an extremely rapid-varying function of $k_e z_o = k_o z_o\sqrt{1-(\omega_p/\omega)^2}$ (the gap width measured in terms of the optical wavelength in the plasma). This follows from the fact that $\tilde{Z} = k_e z_o/\delta$ and $\delta \to 0$. Thus, the conductance of the antenna depends critically on the precise gap width in terms of the optical wavelength.

Next, let us consider the outer limit in (3.10). It may be shown that the expressions in (3.11) and (3.12) are still valid. Now note that the third term in (3.13) behaves as

$$\frac{2}{\pi}\int_0^{\pi/2} \frac{\cos(\delta\ Z\sin\theta)\,d\theta}{(\omega/\omega_p)^2 + \tan^2\theta} = O(\delta) \quad . \tag{3.16}$$

Then M(Z) becomes

$$M(Z) = 1 + 2(s^2 - 1)\cos\frac{sZ}{\delta} + O(\delta) \quad . \text{(outer limit)} \tag{3.17}$$

The second term is a rapidly oscillating term, and M(Z) can assume any value between $(2s^2 - 1)$ and $(3 - 2s^2)$ due to a small (of order δ) variation in $k_e z_o$. However, we note further that the second term in (3.17) is the contribution of the surface wave. When a little loss is introduced, it becomes approximately

$$[2(s^2 - 1)\ \cos\frac{sZ}{\delta}]\ e^{-\tau\frac{sZ}{\delta}}$$

where

$$\tau = \frac{\nu}{\omega}\ \frac{1 + 2(\omega_p/\omega)^2}{1 + (\omega_p/\omega)^2}$$

which is exponetially small and therefore

$$M(Z) = 1 + 0(\delta)$$

for lossy plasma. Thus, an agreement between the warm and cold plasma is obtained.

The conclusion of this antenna problem may be stated as follows. The question of interest is whether the conductance of an infinitely long cylindrical antenna in a warm plasma is a perturbation of that in a cold plasma, and whether these two solutions agree with each other in the zero-temperature limit. Our analysis shows that the answers depend on z_o, the width of the feed gap. If the gap width is small compared with the optical wavelength (so that $k_p z_o$ is fixed), the warm plasma solution cannot be reduced to the cold one when the temperature approaches zero. Furthermore, the conductance in warm plasma is an extremely rapid-varying function of $k_o z_o$, and, therefore, is not a physically well-defined quantity. If the gap is comparable to the optical wavelength and if the plasma has small collision loss, the warm and cold solutions for the conductance then become almost the same. However, in solving antenna problems, $k_o z_o$ is usually assumed to be so small that $k_p z_o$ is fixed. This explains why a variety of different values for the conductance can be obtained, depending on the exact value of $k_o z_o$.

4. THE WARM PLASMA MODEL BASED ON THE KINETIC THEORY

In addition to the (fluid) warm plasma model used in the previous section,* the temperature effect can be also described by the more sophisticated kinetic theory at the expense of the mathematical simplicity. To date, one of the few electromagnetic wave boundary value problems that have been solved by using the kinetic theory is the reflection from a plasma half space. In Section 3, we have examined the solution of the problem based on the fluid theory and have found the existence of the boundary layer in the parameter space for the field solution (e.g. reflection coefficient). In this connection, an interesting question is whether the more sophisticated kinetic theory can rescue us from such a singular behavior in the field solution for a low-temperature plasma.

As in the fluid model, a crucial boundary condition at the plasma free space interface is one regarding the velocity of the electrons. Referring to Figure 2, instead of the rigid condition on the mean velocity of all electrons used in the fluid model

$$V_x\ (x = 0, z) = 0 \tag{4.1}$$

the condition of specular reflection is commonly employed in the kinetic theory, namely,

$$f(v_x, v_z) = f(-v_x, v_z) \quad \text{at } x = 0 \tag{4.2}$$

---

* Alternative names are "transport equation model" and "hydrodynamic thoery."

where f is the perturbed distribution function for the electrons. Recalling that

$$V_x(x = 0, z) = \iint_{-\infty}^{\infty} v_x f(v_x, v_z)\Big|_{x = 0} \, dv_x \, dv_z \quad ;(4.3)$$

the enforcement of (4.2) implies (4.1).

Using the linearized collisionless Boltzman-Vlasov equation and the usual Maxwell equations, the reflection coefficient for an incident H-wave (H_y, E_x, E_z) from a plasma half space subject to the condition in (4.2) is found to be[5,6,7]

$$\Gamma = \frac{Z_o \cos^2\theta_o - Z}{Z_o \cos^2\theta_o + Z} \quad . \tag{4.4}$$

Here $Z_o = \sqrt{\mu_o/\varepsilon_o}$, and Z is the surface impedance of the plasma half space defined by

$$Z = -\frac{E_z}{H_y}\Big|_{x = 0+} \tag{4.5}$$

and is given by

$$Z = \lim_{x \to 0+} \frac{\omega}{i\pi\varepsilon_o} \int_{-\infty}^{\infty} \left[\frac{k_x^2}{c^2 k^4 D_T(k)} - \frac{k_z^2}{\omega^2 k^2 D_L(k)} \right] e^{ik_x x} \, dk_x \tag{4.6}$$

where $\bar{k} = k_x\hat{x} + k_z\hat{z}$, and $k = |\bar{k}|$. The two functions $D_T(k)$ and $D_L(k)$ are plasma dispersion relations for transverse and longitudinal waves, respectively. They are given by

$$D_T(k) = 1 - \left(\frac{\omega}{kc}\right)^2 - \frac{\omega_p^2}{\omega} \int \frac{f_o(v)}{\bar{k} \cdot \bar{v} - \omega} \, d^2v \tag{4.7}$$

$$D_L(k) = 1 - \omega_p^2 \int \frac{f_o(v)}{(\bar{k} \cdot \bar{v} - \omega)^2} \, d^2v \qquad (4.8)$$

where $f_o(v)$ is the unperturbed distribution function for the electrons, and is assumed to be isotropic (depending on $v = \sqrt{v_x^2 + v_z^2}$, not on $\bar{v}$). At this point, it is interesting to mention that if we use the plasma dispersion relations obtained in the fluid model, namely

$$D_T(k) = 1 - \frac{\omega_p^2 - \omega^2}{c^2k^2} \qquad (4.9)$$

$$D_L(k) = 1 - \frac{\omega_p^2}{\omega^2 - a^2k^2} \qquad (4.10)$$

in (4.6), then the integral can be evaluated explicitly to yield the result

$$Z = \sqrt{\frac{\mu_o}{\varepsilon_o}} \frac{1}{\varepsilon} \left[\sqrt{\varepsilon - \sin^2\theta_o} + \frac{(1 - \varepsilon) \sin^2\theta_o}{\sqrt{\frac{\varepsilon}{\delta^2} - \sin^2\theta_o}} \right] \qquad (4.11)$$

Substitution of (4.11) into (4.4) recovers (3.1), as expected.

Return to the solutions obtained by the kinetic theory as given in (4.4) through (4.8). Before trying to evaluate (4.6) for some assumed $f_o(v)$, let us first point out the general features of the present solution which are different from the fluid solution

(i) In addition to zeros, the functions $D_T(k)$ and $D_L(k)$ may have branch cuts in the complex k_x-plane. The location of the possible branch cuts is determined by the condition

$$\bar{k} \cdot \bar{v} - \omega = 0 \quad . \qquad (4.12)$$

Since in the present problem $k_z = k_o \sin\theta_o$, this condition becomes

$$k_x = \frac{\omega - v_z k_o \sin\theta_o}{v_x} \quad . \tag{4.13}$$

If we assume the unperturbed electron velocity cuts off beyond a fixed number v_o

$$f_o(v) = 0, \text{ for } v > v_o \tag{4.14}$$

then we may show that the branch cuts in the complex k_x-plane for a given ω (with $\mathrm{Im}\,\omega > 0$) is determined by two equations (Figure 5)

$$\frac{\mathrm{Im}\, k_x}{\mathrm{Re}\, k_x} = \frac{\mathrm{Im}\,\omega}{\mathrm{Re}\,\omega} \tag{4.15a}$$

$$|k_x| \geq \left| \frac{\omega}{v_o} \sqrt{1 - \left(\frac{v_o}{c}\right)^2 \sin^2\theta_o} \right| = \alpha_o \quad . \tag{4.15b}$$

In evaluating (4.6) by deforming the contour in the upper half k_x plane, we note that the contribution to Z not only includes those from the zeros of D_T and D_L but also from a branch-cut integral. The addition of the branch-cut integral in the kinetic theory accounts for the existence of the van Kampen modes (modes with continuous spectrum) in the plasma half space.

(ii) In the limit $\mathrm{Im}\,\omega \to 0+$, the dispersion functions in (4.7) and (4.8) may assume complex value. Let us concentrate on (4.8), which may be rewritten as

$$D_L(k) = 1 - \omega_p^2 \int_{-v_o}^{v_o} \frac{F_o(v_1)}{(kv_1 - \omega)^2}\, dv_1 \tag{4.16}$$

where (v_1, v_2) are the components of $\bar{v}$ (parallel, perpendicular to $\bar{k}$), and

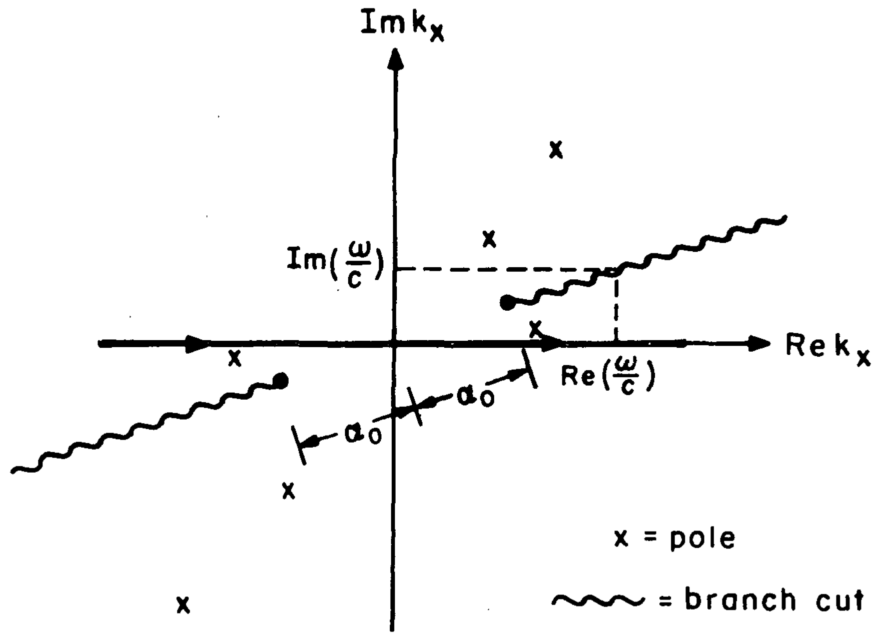

Figure 5. Branch cuts in the complex k-plane for the integrals in (4.7) and (4.8)

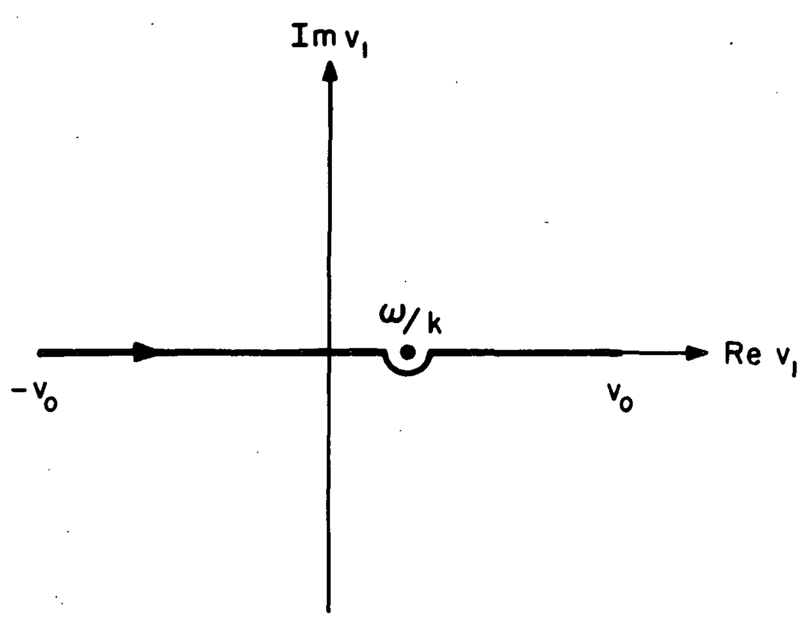

Figure 6. Contour of integration in the complex ν_1-plane for the integral in (4.16)

$$F_o(v_1) = \int_{-\sqrt{v_o^2 - v_1^2}}^{+\sqrt{v_o^2 - v_1^2}} f_o(v)\, dv_2 \quad . \tag{4.17}$$

Referring to Figure 6 the integral in (4.16) can be broken up into a principal value integral and a possible residue contribution

$$D_L(k) = 1 - \omega_p^2 \,⨍_{-v_o}^{v_o} \frac{F_o(v_1)}{(kv_1 - \omega)^2}\, dv_1 + i\pi RH(|k|v_o - \omega) \tag{4.18}$$

where the bar on the integral sign signifies the principal value integral and

$$H(x) = \begin{cases} 1 \quad , & x > 0 \\ 0 \quad , & x < 0 \end{cases}$$

$$R = \text{residue of } [F_o(v_1)/(kv_1 - \omega)^2] \text{ at } v_1 = (\omega/k) \quad .$$

The (positive) imaginary part in (4.18) indicates that the longitudinal wave with wave number $|k| > (\omega/v_o)$ suffers a (Landau) damping even in a collisionless plasma. An interesting consequence of Landau damping is that when $(\omega/\omega_p)^2 < 1$ (cut-off condition for plasma), the surface impedance Z in (4.5) still has a real part and hence the total reflection ($|\Gamma| = 1$) can never be obtained. Similar comments apply to the transverse wave and its dispersion relation.

Whether the above two differences between the kinetic theory and the fluid theory can lead to significantly different field solutions depends largely on the assumed unperturbed distribution function $f_o(v)$. Commonly, the Maxwellian distribution is used for $f_o(v)$, namely

$$f_o(v) = \left(\frac{1}{2\pi a^2}\right) e^{-\frac{v^2}{2a^2}} \tag{4.19}$$

where

$$v = \sqrt{v_x^2 + v_z^2}$$

a = sound speed in the electron gas.

For such a $f_o(v)$, the reflection coefficient Γ given in (4.4) has been approximately evaluated by Weston.[6] His result is given below

$$\Gamma \approx \begin{cases} \Gamma_o + O(\delta), \text{ if } \varepsilon \text{ is fixed} \\ \\ \Gamma_o \dfrac{1 + \delta g_-}{1 + \delta g_+}, \text{ if } \varepsilon \ll 1 \end{cases} \tag{4.20}$$

where Γ_o is the reflection coefficient obtainable by using a cold plasma model and is given explicitly in (3.2), and

$$g_{\pm} = \frac{\sin^2\theta}{[\sqrt{\varepsilon - \sin^2\theta_o} \pm \varepsilon\cos\theta_o]\sqrt{\frac{\varepsilon}{1-\varepsilon} - \delta^2\sin^2\theta_o}} \quad . \tag{4.21}$$

This result is practically* identical to the solution obtained by the field model in Section 3. Thus, the boundary layer also exists in (4.20), the reflection coefficient obtained by using the kinetic theory!

---

* Comparing (4.21) and (3.3) we note that $g_{\pm} = h_{\pm} + O(\varepsilon)$.

5. RADIATION IN WARM PLASMA

The creation of the boundary layers in the field solution of warm plasmas is attributed to the fact of small δ and to the enforcement of the rigid boundary condition. An interesting question in this connection is whether the warm plasma model can give any significantly different result when the rigid boundary condition is not enforced. An important problem belonging to this category is the radiation from a prescribed current source $\bar{J}(\bar{r})$ in an unbounded warm plasma; it will be studied in some detail in this section.

It has been shown by several authors[8,9] that the Fourier transform* of the electric field $\bar{E}(\bar{r})$, denoted by $\bar{E}(\bar{k})$ is related to $\bar{J}(\bar{k})$ through the general relation

$$\bar{E}(\bar{k}) = i\omega\mu_o \left[\frac{k^2\bar{\bar{I}} - \bar{k}\bar{k}\cdot}{k^2 D_T(k)} - \frac{c^2\bar{k}\bar{k}\cdot}{\omega^2 k^2 D_L(k)} \right] \bar{J}(\bar{k}) \quad . \tag{5.1}$$

The above expressions are valid for both (isotropic) cold and warm plasmas (fluid or kinetic theory). For warm plasmas, the explicit forms of D_T and D_L are given in (4.7) and (4.8) for the model based on the kinetic theory, and in (4.9) and (4.10) for the model based on the fluid theory. For the expressions of D_T and D_L in cold plasma, we may simply set the sound speed $a = 0$ in (4.9) and (4.10).

The result of $\bar{E}(\bar{k})$ as computed from (5.1) may be classified into two types, according to the smoothness of $\bar{J}(\bar{r})$. The first type occurs when $\bar{J}(\bar{r})$ is smooth enough so that $\bar{J}(\bar{k})$ is non-zero only when $(\delta k/k_o) << 1$. In such a case $\bar{E}(\bar{k})$ in (5.1) depends on the values of $D_T(k)$ and $D_L(k)$ only

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* The Fourier transform is $F(\bar{k}) = \int_{-\infty}^{\infty} F(\bar{r})\, e^{-i\bar{k}\cdot\bar{r}}\, d^3r$.

in the range

$$(\delta k/k_o) << 1 \quad . \tag{5.2}$$

It may be shown that, under the condition in (5.2),

$$D(k)\Big|_{warm} = D(k)\Big|_{cold} + 0\left(\frac{k\delta}{k_o}\right)^2 \tag{5.3}$$

where D stands for either D_T or D_L. Explicitly we have[10]

cold plasma:

$$D_T(k) = 1 - \left(\frac{k_o}{k}\right)^2 \left(1 - \frac{\omega_p^2}{\omega^2}\right) \tag{5.4}$$

$$D_L(k) = 1 - \left(\frac{\omega_p}{\omega}\right)^2 \quad ,$$

warm plasma (fluid theory):

$$D_T(k) = 1 - \left(\frac{k_o}{k}\right)^2 \left(1 - \frac{\omega_p^2}{\omega^2}\right) \tag{5.5}$$

$$D_L(k) = 1 - \left(\frac{\omega_p}{\omega}\right)^2 \left[1 + \left(\frac{k\delta}{k_o}\right)^2 + \left(\frac{k\delta}{k_o}\right)^4 + - - -\right] \quad ,$$

warm plasma (kinetic theory):*

$$D_T(k) = 1 - \left(\frac{k_o}{k}\right)\left(1 - \frac{\omega_p}{\omega^2}\right)^2 + \frac{\delta^2}{3}\left(\frac{\omega_p}{\omega}\right)^2\left[1 + \left(\frac{k\delta}{k_o}\right)^2 + - - -\right] \tag{5.6}$$

$$D_L(k) = 1 - \left(\frac{\omega_p}{\omega}\right)^2\left[1 + \left(\frac{k\delta}{k_o}\right)^2 + \frac{\delta}{3}\left(\frac{k\delta}{k_o}\right)^4 + - - -\right] \quad .$$

---

* In the formulas in (5.6), $\delta = a/c$ and $a = \sqrt{3KT/m}$ is consistent with those in (5.4) and (5.5).

A study of (5.4) through (5.6) reveals that

$$\bar{E}(\bar{k})\Big|_{warm} = \bar{E}(\bar{k})\Big|_{cold} + 0\left(\frac{k\delta}{k_o}\right)^2 \qquad (5.7)$$

except when

$$1 - \left(\frac{\omega_p}{\omega}\right)^2 = 0(\delta^2) \qquad (5.8)$$

or

$$\left(\frac{\omega_p}{\omega}\right) = 0(1/\delta) \quad . \qquad (5.9)$$

When condition (5.8) is satisfied, $D_L(k)$ for warm plasma may be significantly different from $D_L(k)$ for cold plasma;* while the satisfaction of (5.9) implies a remarkable deviation between the $D_T(k)$ in the kinetic theory model and $D_T(k)$ in the other two models. Except for these two very special cases, we conclude from (5.7) that there is no essential difference between the radiation field produced by a given smooth current source whether a cold or a warm plasma model is used. This conclusion is established on a general term with no particular reference on the current source, as long as it is sufficiently smooth so that (5.2) is satisfied.

A special feature of the kinetic theory model, as has frequently been mentioned in the literature, is the fact that $D_T(k)$ and $D_L(k)$ defined in (4.7), (4.8), and (4.9) have an imaginary part even for real k in a lossless plasma, accounting for the Landau damping effect.* Take

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* Recall the problem of reflection from a plasma half space discussed in Sections 3 and 4. This scattering problem is equivalent to that of a current sheet radiating at the interface. This explains why a boundary layer exists in the field solution when (5.8) is satisfied.

$D_L(k)$ as an example: its imaginary part is given, for real k, by

$$\operatorname{Im} D_L(k) = 3\sqrt{\frac{3\pi}{2}}\left(\frac{\omega_p}{\omega}\right)^2\left(\frac{k_o}{k\delta}\right)^3 \exp\left[-\frac{3}{2}(k_o/\ k\delta)^2\right] . \qquad (5.10)$$

A direct consequence of (5.10) is that when $\omega < \omega_p$, the complex power radiated from the source

$$P = -\int \bar{E}(\bar{r}) \cdot \bar{J}(\bar{r})^* \, d^3r \qquad (5.11)$$

may still have a real part, indicating dissipation in a lossless plasma. However, we must note that when the condition in (5.2) is satisfied, the imaginary part of $D_L(k)$ in (5.10) is exponentially smaller* than its real part in (5.6). Thus, with a lightly lossy plasma [i.e. let $(\omega_p/\omega)^2 \to (\omega_p/\omega)^2\ /\ (1 + i\nu/\omega)$, and $\delta \to \delta/\sqrt{1 + i\nu/\omega}$ with ν being the collision frequency], the effect of the Landau damping is not as important as that of the collision.

The other type of result which may be obtained from (5.1) is the one associated with $\bar{J}(\bar{k})$ which does not cut off fast enough and may assume finite values in the neighborhood of $k = k_o/\delta$. In such a case, $\bar{E}(\bar{k})$ computed from the warm plasma model may be significantly different from the corresponding $\bar{E}(\bar{k})$ computed from the cold plasma. This may be illustrated by the following example.

A popular method in evaluating antenna impedance in plasma

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* It is known that the longitudinal wave, not the transverse wave, suffers Landau damping in a plasma. $D_T(k)$ also has an imaginary part because the relativistic effect has not been included in the Maxwellian distribution in (4.19).

is the so-called "included e.m.f. method" which entails an assumed current distribution and then computes the antenna resistance through the formula

$$R = \frac{-1}{I_o^2} \; Re \int \bar{E}(\bar{r}) \cdot \bar{J}^*(\bar{r}) \; d^3r \qquad (5.12)$$

where I_o is a normalization factor. For the case of a linear antenna with radius a and half length h, $\bar{J}(\bar{r})$ is commonly assumed to be "triangular,"

$$\bar{J}(\bar{r}) = \frac{I_o}{2\pi a} \left[1 - \frac{|z|}{h} \right] \delta(\rho - a) \, \hat{z} \quad ,$$

$$\text{for } |z| \leq h \quad . \qquad (5.13)$$

The Fourier transform of the current is

$$\bar{J}(\bar{k}) = I_o \left[\frac{\sin(kh\cos\frac{\theta}{2})}{(kh\cos\frac{\theta}{2})} \right]^2 J_o(ka \sin\theta) \, \hat{z} \qquad (5.14)$$

where (k, θ, ϕ) are the spherical components of $\bar{k}$, and J_o is the zeroth order Bessel function. The rate of cut off of $\bar{J}(\bar{k})$ for large k depends on the value of h and a. In the following, let us concentrate on two special cases.

$$\text{(A)} \quad k_o h = 0(1) \text{ and } k_o a = 0(\delta^{1/2}) \qquad (5.15)$$

$$\text{(B)} \quad k_o h = 0(\delta^{1/2}) \text{ and } k_o a = 0(\delta^{2/3}) \qquad (5.16)$$

The choice of the above parameters is designed to make the results physically illuminating as well as mathematically simple. For large k such that

$$K = \frac{k\delta}{k_o} \text{ fixed} \tag{5.17}$$

we have

$$\bar{J}(\bar{k}) = \begin{cases} O(\delta^{9/4}), \text{ for case A} & (5.18) \\ \\ O(\delta), \text{ for case B} & (5.19) \end{cases}$$

As we will show next, $\bar{J}(\bar{r})$ in case A is sufficiently smooth so that there is no appreciable difference between the cold and the warm plasma solutions for resistance R in (5.12), but $\bar{J}(\bar{r})$ for case B is not. Under the condition that

$$\frac{k_o \ \varepsilon \ h}{\delta} >> 1 \text{ and } h >> a, \text{ where } \varepsilon = \sqrt{1 - (\omega_p/\omega)^2} \tag{5.19a}$$

an approximate expression for R is found to be

$$R \simeq R_o M \tag{5.19b}$$

where R_o is the resistance for cold plasma and

$$M = 1 + \frac{3\pi(1 - \varepsilon)}{\varepsilon^{3/2}} \left(\frac{1}{k_o h}\right)^3 J_o^2 \left(\frac{k_o a\varepsilon}{\delta}\right) . \tag{5.19c}$$

Then it is a simple matter to verify that

$$M = 1 + O(\delta^{1/2}), \text{ for the case A} \tag{5.20}$$

which is essentially unity. However, for the other case we have

$$M \stackrel{\sim}{\sim} 1 + \frac{6(1 - \varepsilon)}{\varepsilon^{5/2}} \left[\frac{\delta}{(k_o h)^3 k_o a}\right] \cos^2\left(\frac{k_o a\varepsilon}{\delta} - \frac{\pi}{4}\right) \tag{5.21}$$

which is a rapidly oscillating function. If the radius of the antenna

is changed by a small fraction

$$\Delta a = \frac{\pi\delta}{2k_o\,\varepsilon_1} = \frac{\delta}{4\sqrt{1-(\omega_p/\omega)^2}}\,\lambda_o \tag{5.22}$$

where λ_o is the free space wavelength, M in (5.21) may vary from unity to an extremely large number [in the order of $0(\delta^{-7/6})$]. In other words, the acoustic wave can contribute from zero to nearly 100 per cent of the total radiation resistance by a variation of $10^{-3}\lambda_o$ or $10^{-4}\lambda_o$ in radius!

6. CONCLUSION

The warm plasma model based on the fluid theory has been used extensively in the literature. In the last several years there seemed to have been a trend toward treating by this model every electromagnetic boundary value problem previously solved by using the cold plasma model. The primary motivation for doing that is to study the effect of the acoustic wave which is not accounted for in the cold plasma description. However, this effort does not seem to have led to a definite conclusion on the importance of the acoustic wave. Oftentimes, in closely related or even identical problems authors have drawn completely opposite conclusions, depending on their choice of parameters. In the present study, we have discussed an explanation of this, based on boundary layer theory. The main results of our study concerning the (fluid) warm plasma model may be summarized as follows:

(1) In a low-temperature plasma, the acoustic wave constitutes an important part of the total field only in narrow-layered regions close to the rigid boundary. Outside the layers, even a small collision frequently produces an attenuation of the acoustic wave much larger than that of the electromagnetic wave. Therefore, the total field consists almost exclusively of electromagnetic waves.

(2) Due to the excitation of the acoustic waves (dissipated in collision loss or not), the solutions of physical quantities which are of interest in electromagnetic studies are modified. Examples of such physical quantities are the reflection coefficient of an incident electromagnetic wave, radiation power from a current source, input impedance of an antenna, etc. The modification is such that, except

for a few isolated regions in the parameter space forming the so-called "boundary layers," the solutions obtained by using warm and cold plasma descriptions differ only by a negligibly small amount. However, inside the boundary layers the warm plasma solutions vary so rapidly in the parameter space that they cease to be physically meaningful quantities.

(3) The existence of the boundary layers accounts for an important fact that warm plasma solutions may be extremely sensitive to the exact value of the parameters. This perhaps is the main explanation for contradictory conclusions about the importance of the acoustic waves. Thus, in the study of electromagnetic problems by using the warm plasma model, a systematic investigation of the solution dependance on parameters is essential. This is particularly so when the solution is not given in a simple analytical form and numerical computations are necessary. Results based on spot calculations may be very misleading.

(4) Those sensationally different results in the boundary layers of the warm plasma solution probably cannot be observed in practice since the parameters will never be realized exactly enough. Furthermore, the warm plasma model itself perhaps is not a good description of physical plasma when those situations arise.

(5) Another attempt to assess the importance of temperature effects is to use the more exact kinetic theory. The application of this theory is much more complicated and in the few simple cases where it has been worked out, it has not produced significantly new results. In a problem (i.e., reflection from a plasma half space) for which an explicit solution can be found, the solution has practically the same boundary layers in the parameter space as those found in the fluid model. Outside the boundary layers, the solution from the

kinetic theory is again approximately the cold plasma result.

In addition to the boundary value problems, we have also considered the radiation in an unbounded plasma from a given current, a problem in which the rigid boundary condition on the electron velocity is not applied. As long as the current is so smooth that it does not vary significantly over an acoustic wavelength, the radiation field in cold plasma and in warm plasma (fluid or kinetic theory) are practically the same. Furthermore, the often mentioned Landau damping in such cases is not as important as the other loss mechanism such as collision. However, if the current is not smooth, the warm plasma solutions may be significantly different and, in some cases, highly implausible.

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13. ABSTRACT

It is often considered that taking into account the temperature, hence the compressibility, of the electron gas in a plasma is an improvement over the cold plasma model. This however leads to a number of peculiar and paradoxical results that show the need for some caution in applying this model. One result is that the boundary conditions for the warm plasma do not reduce to those for the cold plasma when the temperature approaches zero. Another is that evaluation of the impedance or the radiation from an antenna leads to widely different results according to the exact size of the antenna. This has led some authors to draw completely opposite conclusions as to the importance of acoustic waves. Both of these occurrences can be traced to the fact that the warm plasma equations are of higher order than those of the cold plasma and that the extra terms contain a small factor of the order of a/c, where a is the speed of sound and c that of light.

This suggests that the techniques of the singular perturbation theory can be applied to these problems. Typically the cold plasma model can be applied over wide ranges of the parameters (position, frequencies, angle of incidence) and only over narrow ranges forming so-called "boundary layers" does one need to use the warm plasma model.

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Then again simplified equations can be used and the solutions matched on both sides of the layer's boundary. A number of simple examples will illustrate this point of view. The analysis confirms that some results are highly sensitive to the values of some parameters: wire radius or gap size for an antenna, temperature of the medium, and incident angle of a plane wave. As a result, the corresponding "resonances" cannot be observed in practice since the parameters will never be realized exactly enough. The boundary layer can then be neglected.

The description of the temperature effect by using the more exact kinetic theory is also discussed. The application of this theory is much more complicated, and in a few simple cases where it has been worked out, no significantly new result has been obtained.

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